Engineering Notes

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Stability of a Precision Attitude Determination Scheme

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Introduction

A PRECISION attitude determination scheme suggested for advanced spacecraft makes use of rate integrating gyroscope (RIG's) for measuring spacecraft attitude and star sensors for periodic estimation of drift rates of the RIG's by means of a discrete Kalman filter algorithm. Farrenkoff[1] has worked out the generalized results of such an attitude estimator and also the closed-form solution[2] of the steady-state covariance matrices. The drift rate model of the RIG in such algorithms consists[1-3] of a rate bias b, a white noise drift rate η_v, and a random walk drift rate η_u. If left uncorrected, the gryo drift will result in gradual error buildup at the output, causing erroneous measurement and control of spacecraft attitude. A fixed amount of drift rate correction based on preflight calibration of the gyro is not useful for a long-period mission as the drift rate varies significantly over a prolonged period of time. These problems are solved by feeding back the periodically updated drift rate to the gyro torquer.[4]

The estimation and correction of the drift rate of the RIG can be formulated as a stochastic optimal control problem consisting of an infinite-time regulator and a Kalman estimator. The stability of such a system is investigated here.

Regulator Problem

The RIG output equations (for single-axis attitude determination) are

$$\begin{bmatrix} \dot{\psi} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ d \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \eta_v \\ \eta_u \end{bmatrix} \qquad (1)$$

or

$$\dot{x} = \bar{A}x + \bar{B}u + w \qquad x = [\psi \;\; d]'$$

which can be discretized as

$$x_{i+1} = Ax_i + Bu_i + Gw_i \qquad (2)$$

where

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} T \\ 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3)$$

and where T is the attitude update interval and w_i the equivalent discrete white noise.

The control u is to be such that the drift rate d remains as small as possible. The RIG output ψ is not sought to be minimized as it is also due to any angular velocity input to the RIG. Thus, u is selected to minimize the quadratic cost function

$$J = \frac{1}{2} \sum_{i=1}^{\infty} [x_i'Qx_i + u_i'Ru_i] \qquad (4)$$

where $Q \geq 0$, $R > 0$. The optimal control is given by[5,6]

$$u_i^* = -Cx_i \qquad (5)$$

where

$$C = (R + B'KB)^{-1}(B'KA) \qquad (6)$$

and where K is the solution of the steady-state, discrete Riccati equation

$$K = (A'KA + Q) - (B'KA)'(R + B'KB)^{-1}(B'KA) \qquad (7)$$

For the given A and B matrices and $R = [1]$, the C matrix takes the form

$$C = \frac{1}{1 + T^2K_{11}} [TK_{11} \;\vdots\; T^2K_{11} + TK_{12}] \qquad (8)$$

where K_{11}, K_{22} are the elements of the K matrix.

If the feedback control should include the gyro drift rate alone, then it is required in Eq. (5) that $C = [0 \;\; 1]$, which implies $K_{11} = 0$ and $K_{12} = 1/T$ in Eq. (8). This means that the K matrix as obtained from Eq. (7) is not positive definite. In fact a positive-definite solution of the K matrix does not exist for any $Q \geq 0$, since the matrix pair (A,B) is not controllable.[5] Hence the infinite-time regulator in which the drift rate itself is used as the feedback control is not asymptotically stable.

Estimation Problem

Periodic attitude measurement from the star sensor is used to estimate the spacecraft attitude and the RIG drift rate by means of a Kalman estimator. The state observation equation is given by

$$z_i = Hx_i + v_i \qquad (9)$$

where $H = [1 \;\; 0]$, and v_i is the white Gaussian measurement noise.

The necessary condition for the existence of the Kalman filter gain matrix is that the matrix pair (A,H) is observable.[5] For the given formulation of the problem, this condition is satisfied and the standard discrete Kalman filter can be derived.[1]

System State Covariance

In order to compute the state covariance of the system under combined estimation and control, the discrete time model is taken[5]:

$$x_{i+1} = Ax_i + Bu_i + Gw_i \qquad z_i = Hx_i + v_i$$

$$u_i = -C\hat{x}_i \qquad \hat{x}_i = \bar{x}_i + K_i(z_i - H\bar{x}_i) \qquad (10)$$

where

$$E\{w_i\} = E\{v_i\} = 0$$

$$E\{w_iw_j'\} = Q_w\delta_{ij} \qquad E\{v_iv_j'\} = R_v\delta_{ij} \qquad (11)$$

and K_i is the Kalman filter gain matrix.

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The mean value of the state vector propagates as

$$\bar{x}_{i+1} = [A - BC]\,[\bar{x}_i + K_i(z_i - H\bar{x}_i)] \tag{12}$$

The mean-square value of the state vector is given by

$$\chi_i = \bar{\chi}_i + M_i \tag{13}$$

where

$$\bar{\chi}_{i+1} = [A - BC]\,[\bar{\chi}_i + M_i - P_i]\,[A - BC]' \tag{14}$$

and

$$M_i = E\{(x_i - \bar{x}_i)(x_i - \bar{x}_i)'\} \qquad P_i = E\{(\hat{x}_i - \bar{x}_i)(\hat{x}_i - \bar{x}_i)'\}$$

$$\chi_i = E\{x_i x_i'\} \qquad \bar{\chi}_i = E\{\bar{x}_i \bar{x}_i'\} \tag{15}$$

For the given system of Eqs. (1) and (5),

$$[A - BC] = [I] \tag{16}$$

which gives

$$\bar{\chi}_{i+1} = \bar{\chi}_i + M_i - P_i \qquad \chi_{i+1} = \bar{\chi}_i + M_i - P_i + M_{i+1} \tag{17}$$

When the system matrices A, B, H and the noise covariances Q, R are constant, the filtering process may reach a steady state. Farrenkoff[2] has shown that this indeed is the case, i.e. when $i \to \infty$,

$$M_i \to M_{i+1} \to M_0 \qquad P_i \to P_0 \qquad K_i \to K_0 \tag{18}$$

Hence, from Eqs. (17)

$$\chi_{i+1} - \chi_i = \bar{\chi}_{i+1} - \bar{\chi}_i = M_0 - P_0 \tag{19}$$

and since the prefilter covariance M_0 is larger than the postfilter covariance P_0, the mean-square values of the state variables at update intervals constitute sequences monotonically increasing in time. This gives rise to an instability of the combined estimator-regulator system, with the result that the attitude determination accuracy will degrade over time.

Conclusion

The known procedure of using the drift rate estimate of a rate-integrating gyroscope for drift correction leads to an unstable infinite-time regulator estimation problem. This fact must be considered when finalizing the sensor specifications to be used in the configuration discussed, so that system accuracy remains within the specified bounds during the expected life of the mission.

References

[1] Farrenkoff, R.L., "Generalized Results for Precision Attitude Reference Systems using Gyros," AIAA Paper 74-903, AIAA Mechanics & Control of Flight Conference, 1974.

[2] Farrenkoff, R.L., "Analytic Steady-State Accuracy Solutions for Two Common Spacecraft Attitude Estimators," *Journal of Guidance and Control*, Vol. 1, July-Aug. 1978, pp. 282-284.

[3] Murrel, J.W., "Precision Attitude Determination for Multimission Spacecrafts," AIAA Paper 78-1248, AIAA Guidance and Control Conference, 1978.

[4] Jude, R.J., "System Study of an Inertial Attitude Measurement System for Earth Pointing Satellite; Attitude and Orbit Control Systems," ESA SP-128, Nov. 1977, pp. 111-125.

[5] Bryson, A.E. and Ho, Y., *Applied Optimal Control*, Blaisdell, Waltham, Mass., 1969, Chaps. 12-14.

[6] Hewer, G.A., "An Iterative Technique for the Computation of the Steady-State Gain for the Discrete Optimal Regulator," *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 4, Aug. 1971, pp. 382-384.

Output Ensemble Average of Periodically Excited Linear Time-Varying Systems

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Introduction

IN this Note, we consider a linear time-varying system with a periodical random phase input. One of the properties to describe the output of such system is its mean square ensemble average. The computation of this value is required, inter alia, in the analysis of periodical random phase maneuvers of an airplane evading from a guided missile.[1] In a recent paper,[2] it was shown for long missile flight times (exceeding 8-10 guidance time constants) that the miss distance can be considered as a stationary random variable, and the ensemble average can be computed as a mean square time average.[3-5] For short flight times, where the output is clearly not stationary, only the ensemble average has a meaning. In the past, the analysis of linear time-varying systems driven by a periodical random process was mainly based on Monte Carlo methods.[6]

In this paper, a direct method for computing the mean square ensemble average of the output of such a system is presented. The result can be obtained either analytically or numerically in a single computer run. The considerable reduction in computational effort makes the method very attractive for intensive system analysis. The applicability of the direct method is demonstrated in an example of a missile intercept problem.

Problem Formulation

Consider a linear, single input-single output, casual time-varying system characterized by its impulse response function $g(t,\theta)$. The system input $X(t)$ and output $Y(t)$ are related by

$$Y(t) = \int_0^t g(t,\theta) X(\theta)\,\mathrm{d}\theta \tag{1}$$

The type of input to be considered is a periodical one

$$X(t) = A \sin(\omega t + \phi_i) \tag{2}$$

and the initial phase ϕ_i is a random variable with probability density function.

$$p_{\phi_i}(\beta) = \begin{cases} 1/2\pi & (-\pi \le \beta \le \pi) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In the general case, $Y(t)$ will be a random nonstationary process. The statistical property of interest considered in this paper is the mean square ensemble average of $Y(t)$.

Mean Square Ensemble Average Calculation

Lemma 1: Let a linear time-varying system defined by Eq. (1) be driven by a periodical input with random phase. The mean square ensemble average of the (nonstationary) output

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